Journal of Sound and Vibration (1986) **109**(3), 467–471

ABSORPTION OF ENERGY BY A PUMPED HARMONIC OSCILLATOR

R. K. COLEGRAVE, D. B. L. STEPHENS AND A. KHOSRAVI
Department of Mathematics, King's College, Strand, London WC2R 2LS, England

(*Received* 7 *August* 1985, *and in revised form* 31 *October* 1985)

A pulsating harmonic oscillator of natural frequency ω_0 and mass fluctuation $\gamma(t)=\frac{1}{2}\dot{M}/M=\varepsilon\nu\cos\nu t$ is considered as a model for a parametrically excited system in dynamics, for a pumped electrical circuit or in quantum optics. A Strutt diagram is constructed in the space of the control parameters ε and $\delta=4\omega_0^2/\nu_0^2$, which shows the regions of bounded and unbounded energy absorption.

1. INTRODUCTION

The time-dependent excitation of a dynamical system is usually termed a parametric motion. We shall refer to the periodic excitation of an oscillator as pumping and the result of pumping as a pulsation. The excitation is usually not a direct pumping of the energy, but a pumping of some more accessible physical variable. For instance, a pulsation in an electric circuit may be induced by pumping the capacitance [1]. We propose to pump the mass of an oscillator and examine the resulting pulsation in the energy. The energy transfer may be bounded or unbounded. In the latter case the system acts as an amplifier [1]. A simple example of an amplifier is a child working up the motion of a swing by periodically varying the moment of inertia of the system at twice the natural frequency.

If the equation of motion is linear, and of second order, then Floquet theory [2] shows that a transition from stability to instability, or from bounded to unbounded energy absorption, occurs when one of the normal solutions is periodic. This enables one to plot a stability chart, or Strutt diagram, in the space of the control parameters.

As will be seen, the Mathieu equation

$$\mathrm{d}^2X/\mathrm{d}T^2+(\delta+2\varepsilon\cos 2T)X=0 \tag{1}$$

may be identified with a mass-pumped oscillator if $\delta>0$ and $\varepsilon\ll 1$. The stability of equation (1) was first considered by Strutt [3] and van der Pol and Strutt [4] and has been more recently discussed by McLachlan [5] and by Nayfeh and Mook [2]. The Strutt diagram shows the regions of the $\varepsilon\delta$ plane which correspond to bounded and unbounded solutions of equation (1).

In this note we obtain the Strutt diagram for a Hill equation [6] which describes a pulsating oscillator more fully than equation (1). It may refer to a pumped system in dynamics [2] or in electrical circuits [1] or in quantum optics [7–9], where the electromagnetic field in a cavity may be pumped by irradiation of photons from a reservoir of resonant atoms, each mode of the field being represented by a pulsating harmonic oscillator [9].

0022-460X/86/180467+05 $03.00/0

2. SYSTEM EQUATIONS AND ENERGY VARIATION

The damping of an oscillator may be simulated by exponentially increasing the mass. The system is described, albeit with some controversy [10], by the Kanai-Caldirola Hamiltonian [11, 12]

$$H(q, p, t) = \tfrac{1}{2}p^2/M(t) + \tfrac{1}{2}M(t)\omega_0^2 q^2, \tag{2}$$

where q, p are the conjugate co-ordinate and momentum and the damping coefficient γ is related to the mass by

$$\gamma = \tfrac{1}{2}\dot{M}/M \quad \Rightarrow \quad M(t) = M_0 \exp(2\gamma t). \tag{3}$$

The controversy concerns the failure of equations (2) and (3) to describe the dissipation of energy [13], although the correct equation of motion for a damped oscillator is obtained.

The Hamilton equations are

$$\dot{q} = \partial H/\partial p = p/M(t), \qquad \dot{p} = -\partial H/\partial q = -M(t)\omega_0^2 q. \tag{4}$$

Eliminating the momentum p, one obtains

$$\ddot{q} + 2\gamma(t)\dot{q} + \omega_0^2 q = 0, \qquad \gamma(t) = \tfrac{1}{2}\dot{M}/M. \tag{5}$$

Pumping may be achieved by letting the mass depend on the time in a suitable way [14–17] by analogy with the treatment of damping. In equations (5) $\gamma(t)$ is seen as a variable damping or growth, i.e., as a pumping function, if one chooses a periodic mass such as

$$M(t) = M_0 \cos^2 \nu t, \quad \text{or} \quad M(t) = M_0 \exp(2\varepsilon \sin \nu t). \tag{6, 7}$$

The mass as given in equation (6) leads to an exact solution [15] of equation (5), but a physical shortcoming arises from the periodic vanishing of the mass. The mass as given in equation (7) never vanishes and contains ε, the strength of the pumping, as a control parameter. One should remember that it is not the mass itself that is important, but the pumping function $\gamma(t)$. For the mass given in equation (7)

$$\gamma(t) = \varepsilon\nu \cos \nu t, \tag{8}$$

and this leads to a particularly simple equation of motion

$$\ddot{q} + 2\varepsilon\nu \cos \nu t\, \dot{q} + \omega_0^2 q = 0, \tag{9}$$

which we regard as fundamental for the pumped oscillator.

Let us consider the variation of energy with the time. For any mass $M(t)$ we define, by analogy with (2),

$$L(q, p, t) = \tfrac{1}{2}p^2/M(t) - \tfrac{1}{2}M(t)\omega_0^2 q^2. \tag{10}$$

Then [18]

$$dH/dt = \partial H/\partial t = -2\gamma L, \tag{11a}$$

$$dL/dt = \partial L/\partial t + (\partial L/\partial q)(\partial H/\partial p) - (\partial L/\partial p)(\partial H/\partial q) = -2\gamma H - 2\omega_0^2 S \qquad (S = qp), \tag{11b}$$

$$dS/dt = 2L. \tag{11c}$$

The elimination of L and S leads to the third-order equation

$$\dddot{H} - 2(\dot{\gamma}/\gamma)\ddot{H} + [2(\dot{\gamma}/\gamma)^2 - (\ddot{\gamma}/\gamma) + 4(\omega_0^2 - \gamma^2)]\dot{H} - 4\gamma\dot{\gamma}H = 0, \tag{12}$$

showing that the general connection between mass and energy is complicated. However, let us restrict ourselves to the pumping (8) with $\varepsilon \ll 1$; also for simplicity we shall suppose $S(0)=q(0)p(0)=0$. Then, retaining only terms which are linear in γ or its derivatives and using (11a–c) we may express the initial derivatives of H as multiples of $L(0)$: thus,

$$\dot{H}(0)=-2\gamma(0)L(0), \qquad \ddot{H}(0)=-2\dot{\gamma}(0)L(0), \tag{13a, b}$$

$$\dddot{H}(0)=-2[\ddot{\gamma}(0)-4\omega_0^2\gamma(0)]L(0), \tag{13c}$$

and so on. By using Taylor's theorem it is easily found that

$$H(t)\approx H(0)-2\varepsilon L(0)\sin \nu t \qquad (\nu \gg \omega_0,\ \varepsilon \ll 1), \tag{14a}$$

$$H(t)\approx H(0)-(\varepsilon\nu/\omega_0)L(0)\sin 2\omega_0 t \qquad (\nu \ll \omega_0,\ \varepsilon \ll 1). \tag{14b}$$

For intermediate values of ν/ω_0 the behaviour of $H(t)$ is not so clear, but a pumping resonance occurs when $\nu=2\omega_0$ with exponential gain of energy with time.

In the space of the control variables ε and $\delta=4\omega_0^2/\nu^2$, as in equation (1), one thus sees bounded solutions of equation (9) corresponding to equations (14a, b) at (0, 0), $(0,\infty)$ and an instability cusp at (0, 1) as for the Mathieu equation (1). The stability problem may be seen as one in catastrophe theory [19], an approach which comes into its own if, for instance, equation (9) were replaced by the corresponding pendulum equation and the linear Floquet theory no longer applied.

Equation (9) is taken into the normal form

$$\ddot{Q}+(\omega_0^2+\varepsilon\nu^2\sin\nu t-\varepsilon^2\nu^2\cos^2\nu t)Q=0 \tag{15}$$

by the substitution

$$Q=q\exp(\varepsilon\sin\nu t). \tag{16}$$

A perturbative solution for $\varepsilon \ll 1$ and a rotating-wave approximation for $\nu\approx 2\omega_0$, valid for $\omega_0 t \gg 1$, have been given elsewhere [17], and also an investigation of the effect of a smeared pumping frequency [20]. To compare with equation (1), we write

$$\nu t=2T, \qquad \delta=4\omega_0^2/\nu^2. \tag{17}$$

Then equation (15) becomes

$$\mathrm{d}^2Q/\mathrm{d}T^2+[\delta+4\varepsilon\sin 2T-2\varepsilon^2(1+\cos 4T)]Q=0, \tag{18}$$

which approximates to a Mathieu equation similar to equation (1) when $\varepsilon \ll 1$. The transformation

$$x=2T-\pi/2, \qquad y=Q \tag{19}$$

takes equation (18) into the form

$$y''+(\delta-2\varepsilon^2+4\varepsilon\cos x+2\varepsilon^2\cos 2x)y=0, \tag{20}$$

which, under the substitution

$$(\delta-2\varepsilon^2, 4\varepsilon, 2\varepsilon^2)\rightarrow(\lambda, \gamma_1, \gamma_2), \tag{21}$$

coincides with an equation considered by Klotter and Kotowski [21]. Stability charts in (λ, γ_1) space with $\gamma_2=0$ (the Mathieu case) and $\pm 0{\cdot}5$, together with some further tables have been given in reference [21]. The ranges $\lambda=(-0{\cdot}5, 1{\cdot}5)$, $\gamma_1=(0, 1{\cdot}5)$ are rather restricted. For the pulsating oscillator one needs to use the control parameter δ given by the second of equations (17) rather than λ given by equation (21) and this, coupled with the fact that the parameter γ_2 is not sufficiently varied, means that the computations given in reference [21] are not helpful in the present study.

3. STABILITY ANALYSIS AND CONCLUSIONS

A procedure for the stability analysis of an equation such as equation (18) has been discussed fully in Chapter 5 of reference [2]. Linearly independent solutions u_1, u_2 of equation (18) satisfying the initial conditions

$$u_1(0)=1, \quad \dot{u}_1(0)=0 \quad \text{and} \quad u_2(0)=0, \quad \dot{u}_2(0)=1 \tag{22}$$

are used to construct

$$\alpha(T)=\tfrac{1}{2}[u_1(T)+\dot{u}_2(T)]. \tag{23}$$

The transition curves between regions of stability and instability correspond to periodic solutions of period π if $\alpha=+1$, or period 2π if $\alpha=-1$. Equation (18), like equation (1), clearly has $\alpha=-1$ curves starting at $\varepsilon=0$, $\delta=1^2, 3^2, 5^2, \ldots$, and $\alpha=+1$ curves starting at $\varepsilon=0$, $\delta=2^2, 4^2, 6^2, \ldots$. A computer program was written to plot the curves and it reproduced the known Strutt diagram [2, 5] for the Mathieu equation (1). The results of the computation are shown in Figure 1. Figure 2 is an enlargement in the neighbourhood

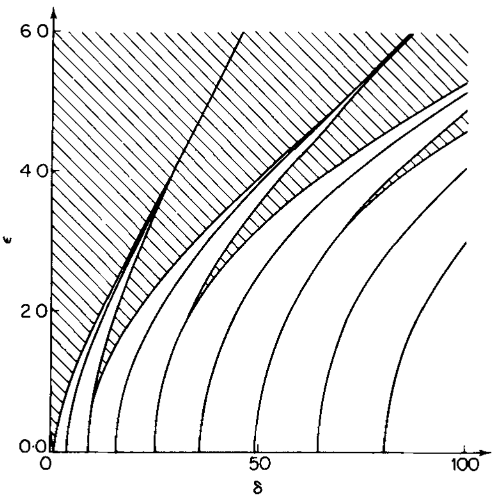

Figure 1. Regions of bounded and unbounded (shaded) energy absorption.

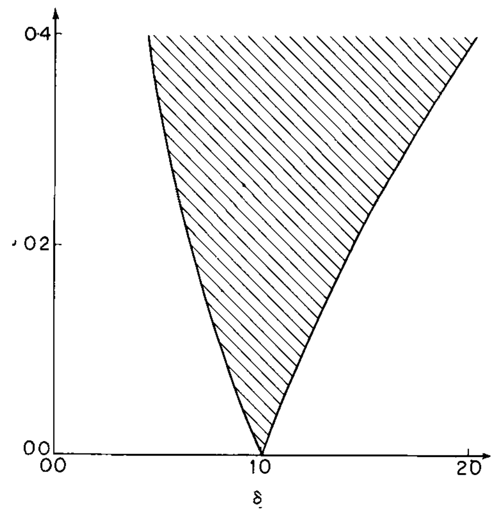

Figure 2. Regions of bounded and unbounded (shaded) energy absorption in the neighbourhood of $\delta=1$.

of the cusp at $\delta = 1$. The regions of instability, or gain of energy, are cross-hatched. The $\alpha = \pm 1$ curves are not necessarily boundaries between regions of bounded and unbounded energy. They may be merely isolated lines along which periodic solutions of equation (18) co-exist.

Let us make a brief comparison between Figure 1 for equation (18) and the Strutt diagram for equation (1). The transition curve $\alpha = -1$ starting at the cusp (0, 1) approaches the ε axis asymptotically, as it should do, rather than running into the unphysical $\delta < 0$ region. All the $\alpha = \pm 1$ asymptotic lines for equation (1) are rotated clockwise in the chart for equation (18). At a sufficiently low frequency ν (so that δ is large) the effect is to give a number of alternate intervals of bounded and unbounded energy absorption when the pumping strength ε is increased; e.g., when $\delta = 40$ the absorption is bounded for $\varepsilon \lesssim 2{\cdot}5$, unbounded for $2{\cdot}5 \lesssim \varepsilon \lesssim 2{\cdot}7$, bounded for $2{\cdot}7 \lesssim \varepsilon \lesssim 3{\cdot}3$, unbounded for $\varepsilon \gtrsim 3{\cdot}3$. However, the most striking difference between the Mathieu equation (1) and the Hill equation (18) is that for the latter the $\alpha = +1$ curves do not bifurcate to include instability regions. They are squeezed between the $\alpha = -1$ instability regions with no chance of opening out. Such co-existence is discussed in reference [6].

As we have seen, the general connection between mass and energy pulsation is complicated. Only in the "good" case $\nu \gg \omega_0$, $\varepsilon \ll 1$ does the energy fluctuate with the mass frequency ν, as shown in equation (14a). Perhaps a fluctuation function $\gamma(t) = \frac{1}{2}\dot{M}/M$ could be found to make $H(t)$ periodic with a given frequency, but this is a difficult inverse problem. Our mass pumping function (8) has the advantage of simplicity and, in the case $\varepsilon \ll 1$, it describes a circuit with weakly pumped capacitance [1]. Similarly it describes the child on a swing before the oscillations become too large, or the situation in quantum optics of a cavity field pumped by an atomic reservoir [9].

REFERENCES

1. W. H. Louisell 1960 *Coupled Mode and Parametric Electronics.* New York: John Wiley.
2. A. H. Nayfeh and D. T. Mook 1979 *Nonlinear Oscillations.* New York: John Wiley. See pp. 260, 273, 278.
3. M. J. O. Strutt 1928 *Massenverteilung Annalen Physik* **85**, 145.
4. F. van der Pol and M. J. O. Strutt 1928 *Philosophical Magazine* **5**, 18.
5. N. W. McLauchlan 1947 *Theory and Application of Mathieu Functions.* Oxford University Press.
6. W. Magnus and S. Winkler 1966 *Hill's Equation.* New York: John Wiley.
7. R. K. Colegrave and M. S. Abdalla 1981 *Optica Acta* **28**, 495.
8. R. K. Colegrave and M. S. Abdalla 1983 *Optica Acta* **30**, 861.
9. R. K. Colegrave and A. Vahabpour-Roudsari 1985 *Optica Acta* **32**, 485.
10. J. R. Ray 1979 *American Journal of Physics* **47**, 626.
11. E. Kanai 1948 *Progress of Theoretical Physics* **3**, 440.
12. P. Caldirola 1983 *Nuovo Cimento* **77B**, 241.
13. R. K. Colegrave and E. Kheyrabady 1984 *Lett. Nuovo Cimento* **40**, 185.
14. B. Remaud and E. S. Hernandez 1980 *Journal of Physics A: Mathematics General* **13**, 2013.
15. R. K. Colegrave and M. S. Abdalla 1982 *Journal of Physics A: Mathematics General* **15**, 1549.
16. M. S. Abdalla and R. K. Colegrave 1985 *Physical Review A* **32**, 1958.
17. M. S. Abdalla, R. K. Colegrave and A. Khosravi 1985 *Nuovo Cimento B* (to appear).
18. H. Goldstein 1980 *Classical Mechanics.* Reading, MA: Addison-Wesley. See p. 256.
19. P. T. Saunders 1980 *An Introduction to Catastrophe Theory.* Cambridge University Press.
20. R. K. Colegrave and A. Khosravi 1986 *Journal of Sound and Vibration.* The effects of uncertain frequency and phase in resonance vibrations.
21. K. Klotter and G. Kotowski 1943 *Zeitschrift fur angewandte Mathematik und Mechanik* **23**, 149–155.