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THE EFFECT OF DAMPING ON A PUMPED HARMONIC OSCILLATOR

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The previous study of a harmonic oscillator of natural frequency ω_0 pumped at frequency ν and intensity ε is extended to include undercritical linear damping at rate γ. Strutt diagrams in the space of the control parameters ε and $\delta=(2\omega_0/\nu)^2$ indicate the extent of the regions of energy absorption when damping is present.

1. INTRODUCTION

The supply of energy to a system in a periodic cycle, either directly or indirectly by some parametric excitation, will be referred to as *pumping*. An idealized pumped oscillator has a displacement $q(t)$ given by the equation [1-3]

$$\ddot{q}+2\varepsilon\nu\cos(\nu t)\dot{q}+\omega_0^2 q=0, \tag{1}$$

which may be regarded as a generalization of the familiar damped harmonic oscillator equation in which the damping constant $\gamma(>0)$ is replaced by a pumping function $\gamma(t)=\varepsilon\nu\cos\nu t$ of frequency ν and strength ε. Examples of pumped systems described by equation (1), when ε is sufficiently small, are: (a) a pendulum performing small oscillations excited by periodically varying its length; (b) an electric circuit pumped by periodically varying the distance between the plates of a capacitor [4, 5]; (c) an optical frequency converter in which energy is shuttled to and fro between two modes of the radiation field [6, 7]; and (d) the supply and removal of energy in a Rabi cycle when a resonant two-level atom is acted on by an electric field [6]. In all these cases it is important to know whether energy is accepted by the system in such a way that a net cycle-averaged gain occurs. Damping is always present in a real system and one needs to know if the gain overcomes the losses. Equation (1) is easily modified to include damping, assumed to be linear in the velocity. The equation becomes

$$\ddot{q}+2(\varepsilon\nu\cos\nu t+\gamma)\dot{q}+\omega_0^2 q=0. \tag{2}$$

With the transformation

$$Q=q\exp(\varepsilon\sin\nu t) \tag{3}$$

and using the dimensionless time $T=\frac{1}{2}\nu t$ and the frequency parameter

$$\delta=(2\omega_0/\nu)^2, \tag{4}$$

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equation (1) takes the form of a Whittaker-Hill equation

$$d^2Q/dT^2+[\delta+4\varepsilon \sin 2T-2\varepsilon^2(1+\cos 4T)]Q=0, \tag{5}$$

which is known to exhibit coexistence [2, 8]. Strutt diagrams in the plane of the control parameters (ε, δ), showing regions of bounded and unbounded Q, have been obtained in previous communications [2, 9] for various time-dependent oscillators, including the pumped oscillator represented by equation (5). When no damping is included, regions of unbounded Q correspond to regions of long-term energy gain. The indirect investigation of energy absorption in reference [2] and the present paper is useful because direct knowledge of the energy of a time-dependent oscillator is difficult to acquire [1, 10, 11].

The effect of the inclusion of damping, as in equation (2), will be considered in the next section. It will be found that as γ is increased from zero to the critical value ω_0, the regions of unbounded Q are displaced in the direction of increasing δ and at the same time the previous lines of coexistence starting at the points $\varepsilon=0, \delta=2^2, 4^2, 6^2, \ldots$ separate out to give new regions of instability for Q. The actual regions of energy absorption are investigated in section 5 by direct calculation of the periodic solutions of equation (2).

2. THE EFFECT OF DAMPING

The mass $M(t)$ to be used in the Kanai-Caldirola Hamiltonian [12, 13]

$$H(q, p, t)=\tfrac{1}{2}p^2/M(t)+\tfrac{1}{2}M(t)\omega_0^2q^2 \tag{6}$$

for the damped system represented by equation (2) may be found by identifying the pumping function with the logarithmic mass derivative as in reference [2]:

$$\tfrac{1}{2}\dot{M}/M=\varepsilon\nu \cos \nu t+\gamma. \tag{7}$$

Hence the mass is

$$M(t)=M_0 \exp[2(\varepsilon \sin \nu t+\gamma t)]. \tag{8}$$

Hamilton's equations are

$$\dot{q}=\partial H/\partial p=p/M(t), \qquad \dot{p}=-\partial H/\partial q=-M(t)\omega_0^2 q. \tag{9}$$

The equation of motion (2) follows by eliminating p from equations (9), using the mass given by equation (8). When transformation (3) is generalized to

$$Q=q \exp(\varepsilon \sin \nu t+\gamma t), \tag{10}$$

equation (2) becomes

$$\ddot{Q}+[\omega_0^2-\gamma^2+\varepsilon\nu^2 \sin \nu t-2\varepsilon\gamma\nu \cos \nu t-\varepsilon^2\nu^2 \cos^2 \nu t]Q=0. \tag{11}$$

Finally, with $T=\tfrac{1}{2}\nu t$ and differentiation with respect to T denoted by a prime, equation (11) takes the form

$$Q''+\{\lambda+4\varepsilon[\sin 2T-2(\gamma/\nu)\cos 2T]-2\varepsilon^2 \cos 4T\}Q=0, \tag{12}$$

where

$$\lambda=\delta-4(\gamma/\nu)^2-2\varepsilon^2 \tag{13}$$

and δ is given by equation (4). Equation (12) provides the generalization of equation (5) when damping is present.

A stability analysis for Q given by equation (12) follows in the next two sections. Unfortunately, owing to the factor $\exp(-\gamma t)$ when equation (10) is used to express $q(t)$ in terms of $Q(t)$, a long-term gain in energy does not necessarily occur whenever Q is unbounded. However, gain cannot occur in the regions of the Strutt diagram where Q is bounded.

3. CALCULATION OF PERIODIC SOLUTIONS FOR Q

It is well known that the regions of bounded and unbounded solutions of a Hill equation such as equation (12) are separated by lines along which the solutions are periodic [5, 8]. This fact enables one to calculate the boundary curves analytically as Taylor series in ε, a procedure which has been used for the Mathieu equation [5, 14]. However, in the present case the procedure is much more complicated owing to the presence of both even and odd terms in the periodic coefficient of Q in equation (12).

Substituting the perturbational expansions

$$\lambda = \lambda_0 + \varepsilon\lambda_1 + \varepsilon^2\lambda_2 + \cdots, \qquad Q = Q_0 + \varepsilon Q_1 + \varepsilon^2 Q_2 + \cdots, \tag{14a, b}$$

into equation (12) and equating powers of ε leads to

$$Q_0'' + \lambda_0 Q_0 = 0, \tag{15a}$$

$$Q_1'' + \lambda_0 Q_1 = [-\lambda_1 - 4\sin 2T + 8(\gamma/\nu)\cos 2T]Q_0, \tag{15b}$$

$$Q_2'' + \lambda_0 Q_2 = [-\lambda_1 - 4\sin 2T + 8(\gamma/\nu)\cos 2T]Q_1 + (-\lambda_2 + 2\cos 4T)Q_0, \tag{15c}$$

and so on. From equation (15a) solutions of period 2π correspond to $\lambda_0 = k^2$, $k = 1, 3, 5, \ldots$, and solutions of period π correspond to $\lambda_0 = k^2$, $k = 2, 4, 6, \ldots$. In either case it is convenient to use normalized even solutions which satisfy [5]

$$Q_I(0) = 1, \qquad Q_I'(0) = 0, \tag{16a}$$

or normalized odd solutions which satisfy

$$Q_{II}(0) = 0, \qquad Q_{II}'(0) = 1. \tag{16b}$$

When one starts with either the solution $Q_0 = \cos kT$ or the odd solution $Q_0 = \sin kT/k$ of equation (15a) it is found that, provided that $k \neq 1$ (and $k \neq 2$ for λ_2, as will be seen below),

$$\lambda_1 = 0, \qquad \lambda_2 = [2/(k^2-1)][1 + 4(\gamma/\nu)^2] \tag{17}$$

must be chosen in order to exclude secular terms in the solutions of equations (15b) and (15c).

Since the coefficient of Q in equation (12) involves both even and odd functions of T, the continuation of Q according to equations (15b) and (15c) is neither even nor odd. However, provided $k \neq 1$, the continuation of the even Q_0

$$Q_1 = \frac{1}{2(k+1)}\sin(k+2)T - \frac{k^2-2}{k(k^2-1)}\sin kT + \frac{1}{2(k-1)}\sin(k-2)T - \frac{\gamma/\nu}{k+1}\cos(k+2)T - \frac{2(\gamma/\nu)}{k^2-1}\cos kT + \frac{\gamma/\nu}{k-1}\cos(k-2)T \tag{18a}$$

or the continuation of the odd Q_0

$$Q_1 = -\frac{\gamma/\nu}{k(k+1)}\sin(k+2)T + \frac{2\gamma/\nu}{k(k^2-1)}\sin kT + \frac{\gamma/\nu}{k(k-1)}\sin(k-2)T - \frac{1}{2k(k+1)}\cos(k+2)T + \frac{1}{k^2-1}\cos kT - \frac{1}{2k(k-1)}\cos(k-2)T \tag{18b}$$

give acceptable linearly independent periodic solutions. Similarly, for even or odd Q_0, solutions may be found for Q_2 from equation (15c). However, since the factor $k-2$ appears in denominators, $k=2$ (as well as $k=1$) must be excluded. Similarly $k=3$ (as well as $k=1$ and $k=2$) must be excluded in the general formulae for Q_3 derived from even or odd Q_0, and so on.

In the case $k=1$, corresponding to the primary resonance, it is found that whether one starts with $Q_0=\cos T$ or $Q_0=\sin T$ it is impossible to choose λ_1 in such a way that secular terms in the solution of equation (15b) are excluded, since both $\cos T$ and $\sin T$ occur when the right side is expressed as the sum of single trigonometrical ratios. It is necessary to work with linear combinations of the perturbed "even" and "odd" solutions, when equation (15b) is found to have the general solution

$$\begin{aligned} Q_1 = \sigma[&\alpha \cos T+\beta \sin T+\tfrac{1}{4}\sin 3T-(\gamma/2\nu)\cos 3T \\ &+\{2(\gamma/\nu)-\tfrac{1}{2}\lambda_1\}T\sin T+T\cos T] \\ &+\tau[\alpha'\cos T+\beta'\sin T-(\gamma/2\nu)\sin 3T-\tfrac{1}{4}\cos 3T \\ &-T\sin T+\{2(\gamma/\nu)+\tfrac{1}{2}\lambda_1\}T\cos T]. \end{aligned} \tag{19}$$

Periodic solutions are those for which

$$\{2(\gamma/\nu)-\tfrac{1}{2}\lambda_1\}\sigma-\tau=0, \qquad \sigma+\{2(\gamma/\nu)+\tfrac{1}{2}\lambda_1\}\tau=0, \tag{20a, b}$$

and λ_1 is found for consistency. λ_2 is found in a similar way from equation (15c). The results, to replace the equations (17) when $k=1$, are

$$\lambda_1=\pm 2[1+4(\gamma/\nu)^2]^{1/2}, \qquad \lambda_2=-\tfrac{1}{2}-2(\gamma/\nu)^2. \tag{21}$$

When $k=2$, Q_1 is given by equation (18a) or (18b) and $\lambda_1=0$ as in equations (17), but Q_2 and λ_2 have to be found by a separate calculation similar to that for $k=1$. Equations (17) must be modified to read

$$\lambda_1=0, \qquad \lambda_2=\tfrac{2}{3}+\tfrac{8}{3}(\gamma/\nu)^2\pm 4(\gamma/\nu)[1+(\gamma/\nu)^2]^{1/2}. \tag{22}$$

It should be noted that co-existence ceases when $\gamma>0$.

When $k=3$, λ_1 and λ_2 are given by equation (17), but a special treatment involving a rather laborious calculation is necessary to find λ_3. The $\lambda(\varepsilon)$ coefficients to $O(\varepsilon^3)$ are then

$$\lambda_1=0, \qquad \lambda_2=\tfrac{1}{4}[1+4(\gamma/\nu)^2], \qquad \lambda_3=\pm\tfrac{1}{8}[1+4(\gamma/\nu)^2][9+4(\gamma/\nu)^2]^{1/2}. \tag{23}$$

With much labour the process of calculation of $\lambda(\varepsilon)$ could be continued. When $k=n$ one should find $\lambda_1, \lambda_2, \ldots, \lambda_{n-1}$ uniquely from general formulae based on even and odd Q_0. The calculation of λ_n requires linearly independent combinations of solutions and gives two values of λ_n, which prevents the co-existence of solutions for any values of k.

4. CONSTRUCTION OF STRUTT DIAGRAMS

From the calculations described in section 3 one can plot the boundary lines $\delta_\pm(\varepsilon)$ in the Strutt diagram showing the regions of unbounded Q. With $\Gamma=\gamma/\omega_0$ equations (12) and (13) may be written as

$$Q''+[\delta(1-\Gamma^2)+4\varepsilon(\sin 2T-\Gamma\delta^{1/2}\cos 2T)-2\varepsilon^2(1+\cos 4T)]Q=0, \tag{24}$$

$$(1-\Gamma^2)\delta_\pm=\lambda_\pm+2\varepsilon^2. \tag{25}$$

From equations (21)-(23) and (25) it follows that the boundary curves are

$$k=1:\ \delta_\pm=(1-\Gamma^2)^{-1}\pm 2\varepsilon(1-\Gamma^2)^{-3/2}+\tfrac{3}{2}\varepsilon^2(1-\Gamma^2)^{-2}+O(\varepsilon^3), \tag{26a}$$

$$k=2:\ \delta_\pm=4(1-\Gamma^2)^{-1}+\tfrac{4}{3}\varepsilon^2(2\pm 3\Gamma)(1-\Gamma^2)^{-2}+O(\varepsilon^3), \tag{26b}$$

$$k=3:\ \delta_\pm=9(1-\Gamma^2)^{-1}+\tfrac{9}{4}\varepsilon^2(1-\Gamma^2)^{-2}\pm\tfrac{3}{8}\varepsilon^3(1+8\Gamma^2)(1-\Gamma^2)^{-5/2}+O(\varepsilon^4). \tag{26c}$$

The widths $\Delta\delta=\delta_+-\delta_-$ of the regions of unbounded Q, to lowest order in ε, are

$$k=1:\ \Delta\delta=4\varepsilon(1-\Gamma^2)^{-3/2}, \tag{27a}$$

$$k=2:\ \Delta\delta=8\varepsilon^2\Gamma(1-\Gamma^2)^{-2}, \tag{27b}$$

$$k=3:\ \Delta\delta=\tfrac{3}{4}\varepsilon^3(1+8\Gamma^2)(1-\Gamma^2)^{-5/2}. \tag{27c}$$

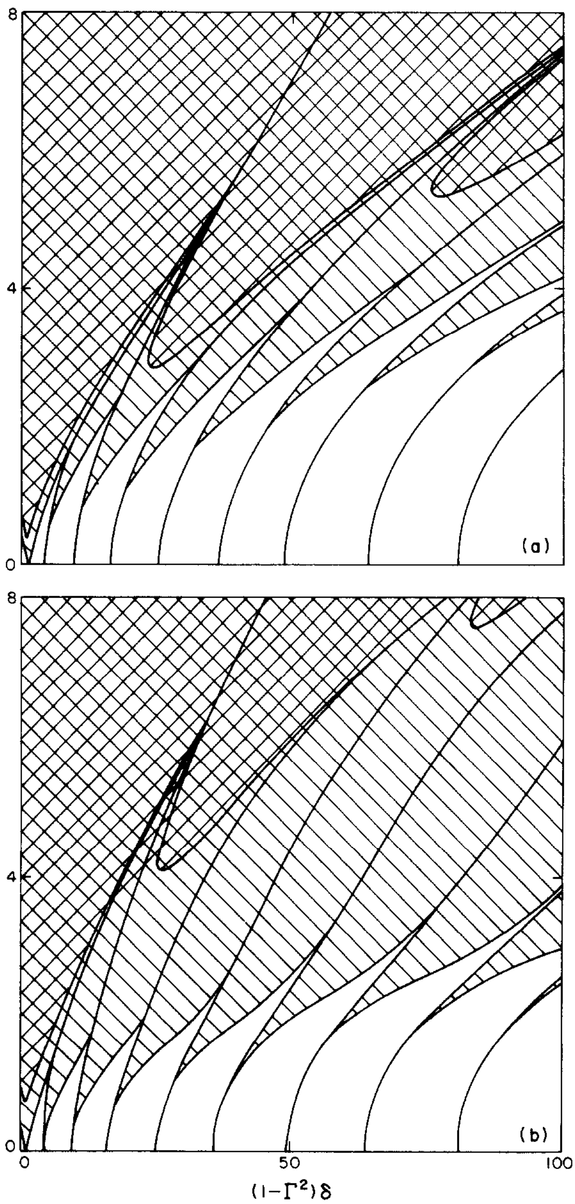

Figure 1. The boundary curves $\delta_\pm(\varepsilon)$ for (a) $\Gamma=0{\cdot}25$ and (b) $\Gamma=0{\cdot}50$.

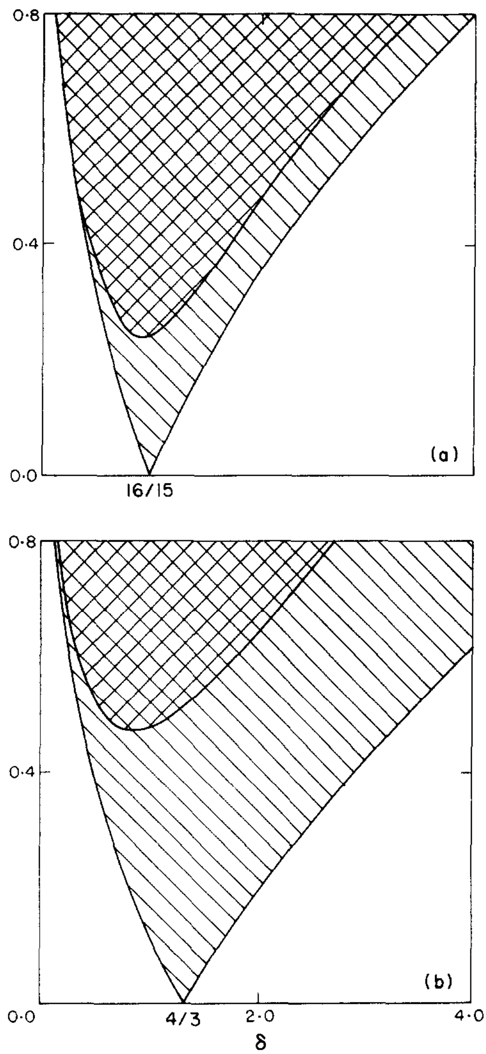

Figure 2. Enlargements of Figures 1(a) and 1(b), respectively, at the principal resonance $\nu = 2\omega$.

Similarly for $k = n$, $\Delta\delta = O(\varepsilon^n)$. Equation (26a) indicates that the width given by equation (27a) is correct to order ε^3.

The stability analysis described in references [2, 9] and in Chapter 5 of reference [5] has been used to obtain a computer plot of the boundary curves $\delta_{\pm}(\varepsilon)$ when $\Gamma = 0{\cdot}25$ and $0{\cdot}50$. The results are shown in Figure 1. For ease of comparison ε is plotted against $(1-\Gamma^2)\delta$. The enlargements in Figure 2 show the changes in the stability region at the principal resonance $\nu = 2\omega$, where $\omega^2 = \omega_0^2 - \gamma^2$, when the damping is increased.

5. DIRECT DETERMINATION OF THE ABSORPTIVE REGIONS

Periodic solutions of equation (2) do not exist for small values of ε and consequently the method described in section 3 does not apply. However, one can still make a computer plot of the transition curves as described in section 4. These curves are the true boundaries of the energy absorbing regions.

Using the dimensionless variables $T=\frac{1}{2}\nu t$, $\delta=(2\omega_0/\nu)^2$ and $\Gamma=\gamma/\omega_0$ introduced earlier, equation (2) takes the form

$$q''+2(2\varepsilon\cos 2T+\Gamma\delta^{1/2})q'+\delta q=0, \tag{28}$$

where the prime denotes differentiation with respect to T. Any solution of equation (28) may be written as

$$q(T)=c_1u_1(T)+c_2u_2(T), \tag{29a}$$

where the fundamental solutions u_1 and u_2 are chosen so that

$$u_1(0)=1, \qquad u_1'(0)=0, \qquad u_2(0)=0, \qquad u_2'(0)=1 \tag{29b}$$

One can follow reference [5] and introduce a matrix A so that $\mathbf{u}(T+\pi)=A\mathbf{u}(T)$: i.e.,

$$u_1(T+\pi)=u_1(\pi)u_1(T)+u_1'(\pi)u_2(T), \qquad u_2(T+\pi)=u_2(\pi)u_1(T)+u_2'(\pi)u_2(T). \tag{30a, b}$$

Then one concentrates on solutions v_1 and v_2 with the property

$$v_1(T+\pi)=\lambda_1v_1(T), \qquad v_2(T+\pi)=\lambda_2v_2(T), \tag{31}$$

where the magnification factors λ_1 and λ_2 are given by

$$|A-\lambda I|=0. \tag{32}$$

Equation (32) reduces to [5]

$$\lambda^2-2\alpha\lambda+\Delta=0, \tag{33}$$

where

$$\alpha=\tfrac{1}{2}[u_1(\pi)+u_2'(\pi)], \qquad \Delta=u_1(\pi)u_2'(\pi)-u_1'(\pi)u_2(\pi). \tag{34a, b}$$

α has to be computed numerically, while Δ is easily calculated to have the value

$$\Delta=\exp(-2\pi\Gamma\delta^{1/2}). \tag{35}$$

Periodic solutions correspond to $\lambda=1$ (period π) or $\lambda=-1$ (period 2π). The boundaries between growing and decaying solutions are given by

$$\alpha=\tfrac{1}{2}(1+\Delta) \qquad (\lambda_1=1, \lambda_2=\Delta), \tag{36a}$$

or

$$\alpha=-\tfrac{1}{2}(1+\Delta) \qquad (\lambda_1=-1, \lambda_2=-\Delta). \tag{36b}$$

The computer program used previously can now be employed to obtain the true energy-absorbing regions, as shown in Figures 1 and 2.

6. DISCUSSION

A study has been made of the effect of undercritical damping on the stability of a pumped oscillator of natural frequency ω_0 performing a motion described by equation (2) or by equation (11). A direct stability analysis of equation (2) is needed to settle the question of energy absorption. Clearly, the instability regions for equation (2) must lie within the corresponding regions for equation (11). In the undamped case [2] the regions coincide. The undercritical damping is represented by the dimensionless parameter

$\Gamma = \gamma/\omega_0$, where $0<\Gamma<1$. The pumping frequency ν is related to the dimensionless parameter δ by equation (4). The intensity of the pumping is represented by ε. Equations (26) show that the cusps in the $(\varepsilon\delta)$ stability chart for equation (12) move from their $\Gamma=0$ positions at $\delta=\kappa^2$ or $\nu=2\omega_0/\kappa$ ($\kappa=1,2,3,\ldots$) to $\delta=\kappa^2(1-\Gamma^2)^{-1}$ or $\nu=2\omega/\kappa$, where $\omega^2=\omega_0^2-\gamma^2$, when $\Gamma>0$. When $\Gamma\to 1$ (critical damping) the $\delta_\pm(\varepsilon)$ transition curves move off to infinity, leaving the entire $(\varepsilon\delta)$ plane as a stability region. This indicates that energy absorption is impossible when $\Gamma\to 1$, as we should expect.

In the undamped case ($\Gamma=0$) the regions of unbounded Q (the "Q regions") and unbounded q (the "q regions" of energy absorption) coincide. When $\Gamma>0$ the cuspidal Q regions still extend to $\varepsilon=0$, whereas the q regions lie completely within the Q regions and do not reach the δ axis.

Firstly, let us discuss the Q regions. The substantial region at the principal resonance $\nu=2\omega$ and the very narrow cuspidal regions at the subharmonic resonances $\nu=2\omega/k$ ($k=2,3,\ldots$) are shown in Figure 1. The widths of the first two of these subharmonic regions are given by equations (27b) and (27c). All the widths will now be expressed directly in terms of the pumping frequency ν.

At the resonant frequencies

$$\nu_k=2(1-\Gamma^2)^{1/2}\omega_0/k \qquad (k=1,2,3,\ldots) \tag{37}$$

the widths given by equations (27) may be expressed as

$$\Delta\delta_k=4\omega_0^2\Delta(1/\nu^2)\approx -8(\omega_0^2/\nu_k^3)\Delta\nu_k, \tag{38}$$

and hence, by combining equations (37) and (38), one has

$$\Delta\nu_k=\nu_+-\nu_-\approx -(1-\Gamma^2)^{3/2}(\omega_0/k)\Delta\delta_k. \tag{39}$$

From equations (27) and (39) the widths of the first three resonances are

$$k=1\colon |\nu_+-\nu_-|=4\varepsilon\omega_0+O(\varepsilon^3), \tag{40a}$$

$$k=2\colon |\nu_+-\nu_-|=\Gamma(1-\Gamma^2)^{-1/2}\varepsilon^2\omega_0+O(\varepsilon^3), \tag{40b}$$

$$k=3\colon |\nu_+-\nu_-|=\tfrac{1}{36}(1+8\Gamma^2)(1-\Gamma^2)^{-1}\varepsilon^3\omega_0+O(\varepsilon^4). \tag{40c}$$

Although equation (27a) holds to order ε^3, $\Delta\nu$ at the principal resonance given by equation (40a) is independent of Γ only to order ε^2, owing to the approximation in equation (38).

In reference (2) it was found that in the undamped case coexistence ($\delta_+=\delta_-$) occurs at $k=2,4,6,\ldots$. This is seen again in equations (27b) and (40b) when $\Gamma=0$. When $\Gamma>0$ narrow detuning widths exist at all the subharmonic resonances and the widths of the regions increase as ε or Γ is increased. The occurrence of $\delta^{1/2}$ in the coefficient of $\cos 2T$ in equation (24) accounts for the increase in extent of the instability regions are k increases, although the cusps become sharper as may be seen in equations (27) and in Figure 1.

The analytical and computed results agree very well for reasonably small values of ε. For $\Gamma=0{\cdot}5$ and $\varepsilon=0{\cdot}2$ equation (27a) gives $\Delta\delta=9{\cdot}6\times 3^{1/2}/9=1{\cdot}85$ compared with $\Delta\delta=1{\cdot}82$ from Figure 2. Similarly, for other values of Γ it is found that the leading terms in equations (40) give accurate results provided that ε does not exceed $0{\cdot}25$. For such values of ε no inaccuracy accrues from the approximation made in equation (38).

Finally, we discuss the q regions. These are the regions in which a damped pulsating oscillator, described by equation (2), absorbs energy from the pumping agency. The fact that Δ given by equation (35) depends on δ makes the computing problem slightly more difficult than for the Q regions. The results are shown in Figures 1 and 2. Probably all that is of interest from a practical point of view is the region lying above the principal resonance $\nu=2\omega_0(1-\Gamma^2)^{1/2}$. No q regions exist above the even subharmonics and those above the odd subharmonics approach the δ axis when $\Gamma \to 0$. The very narrow cuspidal regions are shown in reference [2]. A similar picture of energy absorption has been obtained recently for an equation similar to (2) [15]. The asymmetric placing of the q regions within the Q regions, to be seen in Figures 1 and 2, is noteworthy.

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REFERENCES

1. M. S. ABDALLA, R. K. COLEGRAVE and A. KHOSRAVI 1986 *Nuovo Cimento* **93B**, 195-216. The pulsating harmonic oscillator.
2. R. K. COLEGRAVE, D. B. L. STEPHENS and A. KHOSRAVI 1986 *Journal of Sound and Vibration* **109**, 467-471. Absorption of energy by a pumped harmonic oscillator.
3. B. REMAUD and E. S. HERNANDEZ 1980 *Journal of Physics A: Mathematical & General* **13**, 2013-2018. Damping of the wave-packet motion in a general time-dependent quadratic field.
4. W. H. LOUISELL 1960 *Coupled mode and parametric electronics.* New York: John Wiley. See p. 95.
5. A. H. NAYFEH and D. T. MOOK 1979 *Non-linear Oscillations.* New York: John Wiley. See pp. 261, 276.
6. M. S. ABDALLA, R. K. COLEGRAVE and A. A. SELIM 1988 *Physica* **A151**, 467-481. The effect of periodic energy exchange in the frequency converter.
7. R. K. COLEGRAVE and A. VAHABPOUR-ROUDSARI 1986 *Optica Acta* **33**, 645-652. Amplification in the parametric frequency converter.
8. W. MAGNUS and S. WINKLER 1966 *Hill's Equation.* New York: John Wiley. See p. 106.
9. L. T. WILLE and J. VENNIK 1985 *Journal of Physics A: Mathematical & General* **18**, 1985-1990. Harmonic oscillator with periodic non-zero mass.
10. R. K. COLEGRAVE and E. KHEYRABADY 1984 *Lettere al Nuovo Cimento* **40**, 185-189. The strangled harmonic oscillator.
11. R. K. COLEGRAVE and E. KHEYRABADY 1986 *Physical Review* **A34**, 3634-3641. Dissipation of energy in the damped harmonic oscillator.
12. E. KANAI 1948 *Progress of Theoretical Physics* **3**, 440-442. On the quantization of dissipative systems.
13. P. CALDIROLA 1983 *Nuovo Cimento* **77B**, 241-262. Quantum theory of nonconservative systems.
14. N. W. MCLACHLAN 1947 *Theory and Applications of Mathieu Functions.* Oxford University Press.
15. E. MERON 1987 *Physical Review* **A35**, 4892-4896. Parametric excitation of multimode dissipative systems.